Dynamics of scalar perturbations in $f(R,T)$ gravity

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In the context of $f(R,T)$ theories of gravity, we study the evolution of scalar cosmological perturbations in the metric formalism. According to restrictions on the background evolution, a specific model within these theories is assumed in order to guarantee the standard continuity equation. Using a completely general procedure, we find the complete set of differential equations for the matter density perturbations. In the case of sub-Hubble modes, the expression reduces to a second-order equation which is compared with the standard (quasi-static) equation used in the literature. We show that for general $f(R,T)$ Lagrangians the quasi-static approximation yields to very different results from the ones derived in the frame of the Concordance ΛCDM model.

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I. INTRODUCTION

It is a well-known fact that modifying the law of gravity renders possible explanations for the acceleration mechanism of the Universe [1]-[2]. However it is far from clear which class of dark energy (DE) theories will finally prevail and all the viable mechanism must be studied very carefully. Whereas the theories explaining the accelerated expansion of the Universe in the framework of the general relativity (GR) [3] are usually dubbed as DE models, theories in the framework of modified gravity are more specifically referred to as modified gravity DE theories.

In this letter we consider a class of modified gravity theories in which the gravitational action contains a general function $f(R,T)$, where R and T denote the Ricci scalar and the trace of the energy-momentum tensor, respectively. This kind of modified gravity was introduced first by Harko *et al.* [4] where some significant results were obtained. In the framework of $f(R,T)$ gravity, some cosmological aspects have been already explored: the reconstruction of cosmological solutions, where late-time acceleration is reproduced was studied in Ref. [5] and the energy conditions analyzed in Ref. [6]. The thermodynamics of Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes has been studied in Ref. [7], and also the possibility of the occurrence of future singularities (see Ref. [8]).

So far, a serious shortcoming in this kind of theory has been the non-conservation of the energy-momentum tensor. In this paper we circumvent this problem by showing that an algebraic function $f(R,T)$ can always be constructed to be consistent with the standard energy-momentum tensor conservation. In the following we shall assume separable algebraic functions of the form $f(R,T) = f_1(R) + f_2(T)$. Within this special choice, the function $f_2(T)$ is obtained by imposing the conservation of the energy-momentum tensor.

Once the cosmological background evolution is known, the following step consists of determining the evolution of matter perturbations. The growing of structures is manifestly dependent on the gravitational theory under consideration. This fact can be used to test alternative theories of gravity in order to find out whether those theories are in agreement with GR predictions [9] and experimental data [10] and break the so-called *degeneracy problem* [11] that some modified gravity theories suffer at the background level.

This sort of work has been developed in the last years but mainly for the $f(R)$ gravity scenarios [12]. In this realm, one of us studied the evolution of scalar cosmological perturbation in the metric formalism within the longitudinal

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gauge [13] proving that when scalar cosmological perturbations are studied, $f(R)$ theories, even those mimicking the standard cosmological expansion, usually provide a different matter power spectrum from that predicted by the ΛCDM model.

Still in the framework of $f(R)$ theories of gravity, Gannouji *et al.* considered linear growth of matter perturbations at low redshifts in some DE models and showed the differences with scalar-tensor theories [14]. For further details about cosmological perturbations within $f(R)$ gravity see [15].

Therefore the dynamics of cosmological scalar perturbations is a powerful tool to constrain the viability of different modified gravity theories, by comparing its density contrast evolution with GR expected features [18, 19].

Nonetheless, no attention has been yet paid to study the density contrast evolution in $f(R,T)$ theories.

For our purpose in this communication, the dynamics of linear perturbations are performed studying the problem of obtaining the exact equation for the evolution of matter density perturbations for $f_1(R)+f_2(T)$ type gravitational Lagrangians. More precisely, we shall assume for simplicity the algebraic function $f_1(R)$ as the Einstein-Hilbert term R and the trace dependent function $f_2(T)$ the one for which the covariant conservation of the energy-momentum is accomplished.

In the so-called quasi-static approximation all the time derivative terms for the Bardeen's potentials are neglected, and only time derivatives involving density perturbations are kept [13, 15]. Let us finally point out that this approximation may remove essential information about the evolution of the first-order perturbed fields [16, 22] and therefore requires careful study when considered.

The paper is organized as follows: in Section II, we briefly review the state-of-the-art of $f(R,T)$ gravity. Section III is devoted to introduce the background cosmological equations for $f(R,T)=f_1(R)+f_2(T)$ models as well as the condition to guarantee standard energy-momentum conservation for such models. Then, Section IV addressed the calculation of the scalar perturbed equations for $f(R,T)=f_1(R)+f_2(T)$ models while Section V deals with the study of the quasi-static approximation for this kind of models. In Section VI we apply our results to two particular models and numerical results are obtained and compared with the ΛCDM model. Finally in Section VII we conclude with the main conclusions of this investigation.

II. $f(R,T)$ GRAVITY THEORIES

Let us start by writing the general action for $f(R,T)$ gravities [4],

$$S == S_G + S_m \;=\; \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\; f(R,T) + \int \mathrm{d}^4x\sqrt{-g}\;\mathcal{L}_m \;, \tag{1}$$

where, $\kappa^2 = 8\pi G$, R is the Ricci scalar and T represents the trace of the energy-momentum tensor, $T = T^{\mu}_{\;\mu}$, while $\mathcal{L}_m$ is the matter Lagrangian. As usual the energy-momentum tensor is defined as,

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \,. \tag{2}$$

Then, by varying the action with respect to the metric field $g^{\mu\nu}$, the field equations are obtained,

$$f_R(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} - \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right) f_R(R,T) = -\left(\kappa^2 + f_T(R,T)\right)T_{\mu\nu} - f_T(R,T)\Theta_{\mu\nu}\;, \tag{3}$$

where the subscripts on the function $f(R,T)$ mean differentiation with respect to R or T, and the tensor $\Theta_{\mu\nu}$ is defined as,

$$\Theta_{\mu\nu} \;\equiv\; g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} = -2T_{\mu\nu} - g_{\mu\nu}\mathcal{L}_m + 2g^{\alpha\beta}\frac{\delta\mathcal{L}_m}{\delta g_{\mu\nu}g_{\alpha\beta}} \;. \tag{4}$$

Note that for a regular $f(R,T)$ function, in absence of any kind of matter, the corresponding $f(R)$ gravity equations are recovered, and consequently the corresponding properties and the well-known solutions for $f(R)$ gravity are also satisfied by $f(R,T)$ theories in classical vacuum (for a review on $f(R)$ theories, see [1]). Moreover, here we are interested to study the behavior of this kind of theories for spatially flat FLRW spacetimes, which are expressed in comoving coordinates by the line element,

$$\mathrm{d}s^2 \;=\; a^2(\eta)\left(\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right)\;, \tag{5}$$

where $a(\eta)$ is the scale factor in conformal time η. Then, the main issue arises on the content of the Universe is given by through the energy-momentum tensor, defined in (2). Since we are interested on flat FLRW cosmologies, the usual

content of the Universe (pressureless matter, radiation,...) can be well described by perfect fluids, which to take the form,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} \ . \tag{6}$$

Here ρ and p are the energy and pressure densities respectively, and u^μ is the four-velocity of the fluid, which satisfies $u_\mu u^\mu = 1$, and in comoving coordinates is given by $u^\mu = (1,\ 0,\ 0,\ 0)$. Since $\mathcal{L}_m = p$, according to the definition suggested in Ref. [4], the tensor (4) yields,

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - p\ g_{\mu\nu} \ . \tag{7}$$

Thus the equations motion become

$$f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R = -\left(\kappa^2 - f_T\right) T_{\mu\nu} + f_T p g_{\mu\nu} \ . \tag{8}$$

where we have dropped the explicit dependences of f in R and T.

It is straightforward to see that the usual continuity equation is not satisfied for the field equations (8), and consequently the covariant derivative of the energy-momentum tensor is not null in general $\nabla_\mu T^{\mu\nu} \neq 0$. In order to obtain the modified continuity equation, let us take the covariant derivative of the equation (8),

$$\nabla^\mu \left[f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R = -\left(\kappa^2 + f_T\right) T_{\mu\nu} - f_T \Theta_{\mu\nu} \right]$$

$$\to f_R \nabla^\mu R_{\mu\nu} + R_{\mu\nu}\nabla^\mu f_R - \frac{1}{2} g_{\mu\nu}(f_R \nabla^\mu R + f_T \nabla^\mu T) - (g_{\mu\nu}\nabla^\mu\Box - \nabla^\mu\nabla_\mu\nabla_\nu) f_R =$$

$$\nabla^\mu \left[-\left(\kappa^2 + f_T\right) T_{\mu\nu} - f_T \Theta_{\mu\nu} \right] \ . \tag{9}$$

Using the identities $\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) = 0$, and $(\nabla_\nu\Box - \Box\nabla_\nu) f_R(R,T) = R_{\mu\nu}\nabla^\mu f_R$, the covariant derivative of the energy-momentum tensor needs to satisfy,

$$\nabla^\mu T_{\mu\nu} = \frac{f_T}{\kappa^2 + f_T} \left[\frac{1}{2} g_{\mu\nu}\nabla^\mu T - (T_{\mu\nu} + \Theta_{\mu\nu})\nabla^\mu \ln f_T - \nabla^\mu \Theta_{\mu\nu} \right] \ . \tag{10}$$

Hence, for a perfect fluid with an equation of state $p = w\rho$, being w a constant, the $0-$component of the covariant derivative (10) turns out to become,

$$\left[\kappa^2 + \frac{w-3}{2} f_T - (1+w) T f_{TT} \right] \dot{T} + 3(1+w) \left[H(\kappa^2 - f_T) - 2 f_{TR}(4H\dot{H} + \ddot{H}) \right] T = 0 \, , \tag{11}$$

where let us remind that $T = T^\mu_{\ \mu} = \rho - 3p$. The last equation differs from the usual continuity equation on the non-null right hand side (r.h.s.). Thus, it may lead to violations of the usual evolution of the different species in the Universe. Nevertheless, in the next section we focus our attention on a model that keeps the usual continuity equation unchanged.

III. $f_1(R) + f_2(T)$ TYPE THEORIES

In this section, we choose the algebraic function $f(R,T)$ to be a sum of two independent functions

$$f(R,T) \ = \ f_1(R) + f_2(T) \, , \tag{12}$$

where $f_1(R)$ and $f_2(T)$, respectively depend on the curvature R and the trace T. The generalized Einstein equations from (8) yield

$$-3\mathcal{H} f'_{1R_0} + 3\mathcal{H}' f_{1R_0} - \frac{a^2}{2} f_{10} \ = \ -\kappa^2 a^2 \rho_0 + (1 + c_s^2)\rho_0 a^2 f_{2T_0} + \frac{a^2}{2} f_{20} \ , \tag{13}$$

$$f''_{1R_0} + \mathcal{H} f'_{1R_0} - (\mathcal{H}' + 2\mathcal{H}^2) f_{1R_0} + \frac{a^2}{2} f_{10} \ = \ -\kappa^2 a^2 c_s^2 \rho_0 - \frac{a^2}{2} f_{20} \ . \tag{14}$$

where the prime holds for the derivative with respect to η, $\mathcal{H} \equiv a'/a$ and the subscript 0 holds for unperturbed background quantities: R_0 denotes the scalar curvature corresponding to the unperturbed metric, ρ_0 the unperturbed energy density, with $f_{10} \equiv f_1(R_0)$, $f_{1R_0} \equiv \mathrm{d}f_1(R_0)/\mathrm{d}R_0$, $f_{20} \equiv f_2(T_0)$, $f_{2T_0} \equiv \mathrm{d}f_2(T_0)/\mathrm{d}T_0$ and $c_s^2 = p_0/\rho_0$. The continuity equation (11) for Lagrangians given by (12) yields

$$\nabla_\mu T^{\mu}_{0\,\nu} = \frac{1}{\kappa^2 - f_{2T_0}} \left[\delta^\mu_\nu \partial_\mu \left(\frac{1}{2} f_{20} + c_s^2 \rho_0 f_{2T_0} \right) + T_{0\,\nu}^{\,\mu} \partial_\mu f_{2T_0} \right] \,, \tag{15}$$

showing explicitly that the energy-momentum tensor is not a priori covariantly conserved in $f(R,T)$ theories. Thus, for these theories, the test particles moving in a gravitational field do not follow geodesic lines. By exploring the equation (15) for $\nu = 0$ component, one gets

$$\rho_0^{'} + 3\mathcal{H}\rho_0(1+c_s^2) = \frac{1}{\kappa^2 - f_{2T_0}} \left[(1+c_s^2)\rho_0 f'_{2T_0} + c_s^2 \rho_0' f_{2T_0} + \frac{1}{2} f'_{20} \right] \,. \tag{16}$$

Note that whether f_2 vanishes (i.e., $f(R)$ theories) or characterizes a non-running cosmological constant, both $f_{2T_0}^{'}$ and f_{2T_0} vanish, and then the continuity equation in these scenarios becomes

$$\rho_0^{'} + 3\mathcal{H} \left(1 + c_s^2\right) \rho_0 = 0 \,. \tag{17}$$

In order to Lagrangians such as (12) consistent with the standard conservation equation, the r.h.s. of (15) has to vanish leading to the differential equation

$$\left(1 + c_s^2\right) T_0 f_{2T_0T_0} + \frac{1}{2} \left(1 - c_s^2\right) f_{2T_0} = 0 \,, \tag{18}$$

where $c_s^2 \neq 1/3$. The general solution of this differential equation reads

$$f_2(T_0) = \alpha T_0^{\frac{1+3c_s^2}{2(1+c_s^2)}} + \beta \,, \tag{19}$$

where α and β are integration constants. In the case of a barotropic equation of state $c_s^2 = 0$, i.e., dust, the model (19) becomes

$$f_2(T_0) = \alpha T_0^{1/2} + \beta \,. \tag{20}$$

This function represents the unique Lagrangian that satisfies the usual continuity equation (17) within the class of models given by expression (12).

IV. PERTURBATIONS IN $f(R,T)$ THEORIES

Let us consider the scalar perturbations of a flat FLRW metric in the longitudinal gauge

$$\mathrm{d}s^2 = a^2(\eta) \left[(1+2\Phi)\mathrm{d}\eta^2 - (1-2\Psi)\mathrm{d}\mathbf{x}^2 \right] \,, \tag{21}$$

where $\Phi \equiv \Phi(\eta, \mathbf{x})$ and $\Psi \equiv \Psi(\eta, \mathbf{x})$ are the scalar perturbations. The components of perturbed energy-momentum tensor in this gauge are given by

$$\hat{\delta} T^0_{\ 0} = \hat{\delta}\rho = \rho_0 \delta \,, \quad \hat{\delta} T^i_{\ j} = -\hat{\delta} p \ \delta^i_{\ j} = -c_s^2 \rho_0 \delta^i_{\ j} \delta \,, \quad \hat{\delta} T^0_{\ i} = -\hat{\delta} T^i_{\ 0} = -\left(1 + c_s^2\right) \rho_0 \partial_i v \,, \tag{22}$$

where v denotes the potential for the velocity perturbations. Using the model (12), the perturbed metric (21) and the perturbed energy-momentum tensor (22), the first order perturbed equations reads

$$\begin{aligned} f_{1R_0} \hat{\delta} G^\mu_\nu + \left(R^{\ \mu}_{0\nu} + \nabla^\mu \nabla_\nu - \delta^\mu_\nu \Box\right) f_{1R_0R_0} \hat{\delta} R + \left[\left(\hat{\delta} g^{\mu\alpha} \right) \nabla_\nu \nabla_\alpha - \delta^\mu_\nu \left(\hat{\delta} g^{\alpha\beta} \right) \nabla_\alpha \nabla_\beta \right] f_{1R_0} \\ - \left[g_0^{\alpha\mu} \left(\hat{\delta} \Gamma^\gamma_{\alpha\nu} \right) - \delta^\mu_\nu g_0^{\alpha\beta} \left(\hat{\delta} \Gamma^\gamma_{\beta\alpha} \right) \right] \partial_\gamma f_{1R_0} = -\left(\kappa^2 - f_{2T_0}\right) \hat{\delta} T^\mu_\nu \\ + \left[\frac{1}{2}(1-c_s^2) f_{2T_0} \delta^\mu_\nu + (1-3c_s^2)(1+c_s^2)\rho_0 f_{2T_0T_0} u^\mu u_\nu \right] \hat{\delta}\rho \,, \end{aligned} \tag{23}$$

where $f_{1R_0R_0} = \mathrm{d}^2 f_1(R_0)/\mathrm{d}R_0^2$, $\nabla^\alpha\nabla_\alpha$ and ∇ holds for the covariant derivative with respect to the unperturbed metric (5). In (23), we have made use of the relation linking the trace to the enery density, $T_0 = \rho_0 - 3p_0 = (1-3c_s^2)\rho_0$, and by the way, $\hat{\delta}T = (1-3c_s^2)\hat{\delta}\rho$. Here the equations of motion at the left hand side of (23) presents a set of fourth-order differential equations. By following the same assumptions, the equation of the perturbations of the continuity equation (15) can be easily obtained, which yields,

$$\nabla_\mu \hat{\delta}T^\mu_\nu + \hat{\delta}\Gamma^\mu_{\mu\lambda}T^\lambda_{0\nu} - \hat{\delta}\Gamma^\lambda_{\mu\nu}T^\mu_{0\lambda} = \frac{1}{(\kappa^2 - f_{2T_0})}\Big\{ f_{2T_0T_0}\hat{\delta}T\nabla_\mu T^\mu_{0\nu}$$
$$+\delta^\mu_\nu\partial_\mu\left(\frac{1}{2}f_{2T_0}\hat{\delta}T + p_0 f_{2T_0T_0}\hat{\delta}T + \hat{\delta}p f_{2T_0}\right) + \partial_\mu(f_{2T_0})\hat{\delta}T^\mu_\nu + T^\mu_{0\nu}\partial_\mu(f_{2T_0T_0}\hat{\delta}T)\Big\} \ . \tag{24}$$

from which the conservation equations in $f(R)$ gravity are obtained when $f_2(\tilde{T})$ is equivalent to the cosmological constant, or vanishes.

For the linearised equation (23), the components (ij), (00), (ii) and $(0i) \equiv (i0)$, where $i, j = 1, 2, 3$, $i \neq j$ in Fourier space, read respectively:

$$\Phi - \Psi = -\frac{f_{1R_0R_0}}{f_{1R_0}}\hat{\delta}R \ , \tag{25}$$

$$\Big[3\mathcal{H}\left(\Phi' + \Psi'\right) + k^2\left(\Phi + \Psi\right) + 3\mathcal{H}'\Psi - \left(3\mathcal{H}' - 6\mathcal{H}^2\right)\Phi\Big] f_{1R_0}$$
$$+\left(9\mathcal{H}\Phi - 3\mathcal{H}\Psi + 3\Psi'\right) f'_{1R_0} = a^2\Big[-\kappa^2\rho_0 + \left(1 - 2c_s^2 - 3c_s^4\right)\rho_0^2 f_{2T_0T_0} + \frac{1}{2}\left(3 - c_s^2\right)\rho_0 f_{2T_0}\Big]\delta \tag{26}$$

$$\Big[\Phi'' + \Psi'' + 3\mathcal{H}\left(\Phi' + \Psi'\right) + 3\mathcal{H}'\Phi + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\Psi\Big] f_{1R_0} + \left(3\mathcal{H}\Phi - \mathcal{H}\Psi + 3\Phi'\right) f'_{1R_0}$$
$$+\left(3\Phi - \Psi\right) f''_{1R_0} = a^2\Big[\kappa^2 c_s^2\rho_0 + \frac{1}{2}\left(1 - 3c_s^2\right)\rho_0 f_{2T_0}\Big]\delta \tag{27}$$

$$\left(2\Phi - \Psi\right) f'_{1R_0} + \Big[\Phi' + \Psi' + \mathcal{H}\left(\Phi + \Psi\right)\Big] f_{1R_0} = -a^2\left(1 + c_s^2\right)\left(\kappa^2 - f_{2T_0}\right)\rho_0 v \ , \tag{28}$$

with

$$\hat{\delta}R = -\frac{2}{a^2}\Big[3\Psi'' + 6\left(\mathcal{H}' + \mathcal{H}^2\right)\Phi + 3\mathcal{H}\left(\Phi' + 3\Psi'\right) - k^2\left(\Phi - 2\Psi\right)\Big] \ . \tag{29}$$

where it is easy to remark that for $f_2(T_0) = 0$ the $f(R)$ equations are recovered [13]. Moreover, for $f_1(R_0) = R_0$, the GR equations are obtained [20]. Now, by using $c_s^2 = 0$, (15) and (24), the energy-momentum tensor conservation renders to the following first order equations

$$\begin{aligned}\delta' - k^2 v - 3\Psi' \ &= \ -\frac{3\mathcal{H}f_{2T_0T_0}\rho_0\delta}{(\kappa^2 - f_{2T_0})^2}\left(\frac{1}{2}f_{2T_0} + \rho_0 f_{2T_0T_0}\right) + \frac{1}{\kappa^2 - f_{2T_0}}\Big[\delta'\left(\frac{1}{2}f_{2T_0} + \rho_0 f_{2T_0T_0}\right) \\ &\quad -3\mathcal{H}\delta\left(\frac{5}{2}\rho_0 f_{2T_0T_0} + \rho_0^2 f_{2T_0T_0T_0} + \frac{1}{2}f_{2T_0}\right)\Big]\end{aligned} \tag{30}$$

and

$$\Phi + \mathcal{H}v + v' \ = \ -\frac{1}{\kappa^2 - f_{2T_0}}\left(\frac{1}{2}f_{2T_0}\delta + 3\mathcal{H}\rho_0 f_{2T_0T_0}v\right) \tag{31}$$

for the temporal and spatial components respectively.

From the previous expressions is clear that for $f_2(T_0) \equiv 0$, the usual conservations equations in $f(R)$ theories (GR in particular) are recovered (see for instance eqns. (21) and (22) in [13]). Note that expression (17) has been used in order to obtain both (30) and (31). After further simplifications, the last two expressions become

$$\delta' - k^2 v - 3\Psi' \ = \ 0 \tag{32}$$

and

$$\Phi + \mathcal{H}v + v' \;=\; \frac{f_{2T_0}}{2(\kappa^2 - f_{2T_0})}\,(3\mathcal{H}v - \delta)\;, \tag{33}$$

that when combined yield

$$\delta'' + \mathcal{H}\left[1 - \frac{3f_{2T_0}}{2\,(\kappa^2 - f_{2T_0})}\right]\delta' + k^2\frac{f_{2T_0}}{2(\kappa^2 - f_{2T_0})}\delta + k^2\Phi - 3\Psi'' - 3\mathcal{H}\left[1 - \frac{3f_{2T_0}}{2\,(\kappa^2 - f_{2T_0})}\right]\Psi' \;=\; 0\,. \tag{34}$$

Hence, the complete set of equations that describes the general linear perturbations for the kind of models considered here, $f(R,T) = f_1(R) + f_2(T)$, have been obtained, which provides the enough information about the behavior of the perturbations within this class of theories, that can be compare with those results in ΛCDM model.

V. EVOLUTION OF SUB-HUBBLE MODES AND THE QUASI-STATIC APPROXIMATION

We are interested in the possible effects on the density contrast evolution once the perturbations enter the Hubble radius in the matter dominated era. In the sub-Hubble limit, i.e., $\mathcal{H} \ll k$, and after having neglected all the time derivative for the Bardeen's potentials Φ and Ψ, the equations (25) and (26) can be combined yielding

$$\Psi \;=\; \Phi\left(\frac{1 + \frac{2k^2}{a^2}\frac{f_{1R_0R_0}}{f_{1R_0}}}{1 + \frac{4k^2}{a^2}\frac{f_{1R_0R_0}}{f_{1R_0}}}\right) \quad , \quad \Phi \;=\; -\frac{1}{2k^2}\left(\frac{1 + \frac{4k^2}{a^2}\frac{f_{1R_0R_0}}{f_{1R_0}}}{1 + \frac{3k^2}{a^2}\frac{f_{1R_0R_0}}{f_{1R_0}}}\right)\left(\kappa^2 - f_{2T_0}\right)\frac{a^2\rho_0}{f_{1R_0}}\delta\,. \tag{35}$$

In addition, the equation (34) in the QS approximation yields,

$$\delta'' + \mathcal{H}\left[1 - \frac{3f_{2T_0}}{2\,(\kappa^2 - f_{2T_0})}\right]\delta' + k^2\frac{f_{2T_0}}{2(\kappa^2 - f_{2T_0})}\delta + k^2\Phi \;=\; 0\,. \tag{36}$$

Then, by using the previous result (35) in the equation (36) one gets

$$\delta'' + \mathcal{H}\left[1 - \frac{3f_{2T_0}}{2\,(\kappa^2 - f_{2T_0})}\right]\delta' + \frac{1}{2}\left[k^2\frac{f_{2T_0}}{(\kappa^2 - f_{2T_0})} - (\kappa^2 - f_{2T_0})\frac{a^2\rho_0}{f_{1R_0}}\left(\frac{1 + 4\frac{k^2}{a^2}\frac{f_{1R_0R_0}}{f_{1R_0}}}{1 + 3\frac{k^2}{a^2}\frac{f_{1R_0R_0}}{f_{1R_0}}}\right)\right]\delta \;=\; 0\,, \tag{37}$$

that can be understood as the quasi-static equation for $f(R,T)$ models of the form (19). By neglecting in (37) the terms $f_2(T_0)$, i.e., paying attention only to $f(R)$ theories, one recovers the usual quasi-static approximation for those theories (see for instance [21], [22] and [23])

$$\delta^{''} + \mathcal{H}\delta^{'} - \left(\frac{1 + 4\frac{k^2}{a^2}\frac{f_{1R_0R_0}}{f_{1R_0}}}{1 + 3\frac{k^2}{a^2}\frac{f_{1R_0R_0}}{f_{1R_0}}}\right)\frac{\kappa^2 a^2\rho_0}{2f_{1R_0}}\delta \;=\; 0 \tag{38}$$

and for GR ($f_1(R_0) = R_0$), the quasi-static equation for δ becomes the well-known expression

$$\delta'' + \mathcal{H}\delta' - 4\pi \mathrm{G}\rho_0 a^2\delta \;=\; 0\,, \tag{39}$$

which is k-independent.

Note that the effect of the $f_2(T_0)$ terms in (37) is twofold: first, the coefficient of δ' gets an extra term that depends on the first derivative of $f_2(T_0)$ with respect to T_0 that in general will be time dependent. Second, the coefficient for δ is also modified by adding a k^2 dependence that is absent the standard quasi-static limit both in GR and in $f(R)$ theories and modifying as well the usual coefficient already present for $f(R)$ theories by a factor $(\kappa^2 - f_{2T_0})$. The k^2-presence may have extraordinary consequences since for $f(R)$ theories it is usually claimed that in the two asymptotic limits (i.e., either GR or $f(R)$ domination), the quasi-static equation is scale independent and only in the transient regime, differences associated to the scale may show up. For the class of $f(R,T)$ theories under study there is always a k^2 term that for deep Sub-Hubble modes will be dominant at any time of the cosmological evolution.

On the other hand, a qualitative analysis taking into account that $\kappa^2 \approx M_P^{-2} \approx (10^{19}\mathrm{GeV})^{-2}$ and $f_{2T_0} \approx \rho_{critical}^{-1/2} \approx (10^{-3}\mathrm{eV})^{-2}$, implies that equation (37) may be simplified yielding

$$\delta'' + \frac{5}{2}\mathcal{H}\delta' + \frac{1}{2}\left[-k^2 + f_{2T_0}\frac{a^2\rho_0}{f_{1R_0}}\left(\frac{1 + 4\frac{k^2}{a^2}\frac{f_{1R_0R_0}}{f_{1R_0}}}{1 + 3\frac{k^2}{a^2}\frac{f_{1R_0R_0}}{f_{1R_0}}}\right)\right]\delta \;=\; 0\,. \tag{40}$$

Furthermore, if now one is interested only in extreme sub-Hubble modes, it is clear that (40) becomes

$$\delta'' + \frac{5}{2}\mathcal{H}\delta' + \frac{1}{2}\left[-k^2 + \frac{4}{3}f_{2T_0}\frac{a^2\rho_0}{f_{1R_0}}\right]\delta \,=\, 0\,, \tag{41}$$

that in this limit and after having considered reasonable gravitational Lagrangian, i.e., not divergent, yields

$$\delta'' + \frac{5}{2}\mathcal{H}\delta' - \frac{1}{2}k^2\delta \,=\, 0\,. \tag{42}$$

The last expression, as well as the intermediate results (40) and (41) prove that gravitational Lagrangians depending on the trace of the energy-momentum tensor and satisfying the usual conservation equation will exhibit a density contrast evolution that is k-dependent for sub-Hubble modes. This fact is in contradiction with the usual assumptions about scale-invariant spectrum of matter perturbations before entering the Hubble horizon that can be seen, for instance in the GR density contrast evolution (39). Therefore, models such as (20), i.e., the only ones guaranteeing the standard conservation equation, seem to be theoretically excluded as will be explicitly shown in the next section.

In addition, note that the equation (37) exhibits a singular point at $\kappa^2 - f_{2T_0} = 0$. For the Lagrangian $f_2(T_0) = \alpha T_0^{1/2} + \beta$, such singular point is easily identified. From now on, let's assume the following coupling constant,

$$\alpha \,=\, c_1\kappa^2\rho_0(\eta_{today}))^{1/2}\ , \tag{43}$$

where $\rho_0(\eta_{today}) = \frac{3}{\kappa^2}\Omega_m^0 H_0^2$ and c_1 is a dimensionless constant, whereas Ω_m^0 and H_0 hold for the matter density and the Hubble parameter values today (at redshift $z = 0$) respectively. This parametrization is justified in order to fix the correct dimensions for the coupling constant α. On the other hand, by solving the continuity equation (17) for a pressureless fluid, the evolution of the matter density yields,

$$\rho_0 = \rho_{today}a^{-3} = \rho_{today}(1+z)^3\ , \tag{44}$$

where the usual relation $1 + z = a^{-1}$ has been used. Then, the expression of the denominator in the equation (37) is given by,

$$\kappa^2 - f_{2T_0} = \kappa^2\left[1 - \frac{c_1}{(1+z)^{3/2}}\right]\ . \tag{45}$$

Hence, a singularity occurs at $z_s = c_1^{2/3} - 1$. Then, the avoidance of such singularity constrains the value of the free parameter c_1:

- $c_1 < 0$, the singular point is located at $z_s < -1$, outside of the allowed range for the redshift, as defined above.
- $c_1 > 0$, here we can distinguish between two cases: if $0 < c_1 < 1$, then $-1 < z_s < 0$, and the singularity will occur in the future, while if $c_1 \geq 1$, the singularity is located at $z_s \geq 0$, so at the present or past cosmological evolution.

In order to avoid any singularity, at least for the range $z > 0$, we shall assume $c_1 < 1$. Note also that in the neighbourhood of the singularity, the equation (37) reduces to,

$$\delta'' - \mathcal{H}\frac{3f_{2T_0}}{2\left(\kappa^2 - f_{2T_0}\right)}\delta' + k^2\frac{f_{2T_0}}{2(\kappa^2 - f_{2T_0})}\delta \,=\, 0 \tag{46}$$

and in consequence the perturbations would behave as a damped oscillator, as is analyzed in the following section and shown in Fig. 3.

VI. NUMERICAL RESULTS

In order to check the results obtained in the previous section, we study two particular $f(R,T)$ models where we have assumed that the $f_1(R_0)$ function is represented by the usual term proportional to the Ricci scalar, i.e., $f_1(R_0) = R_0$. This choice encapsulates a modification to GR purely originated by the function $f_2(T_0)$ introduced in Section 2 through the expression (19).

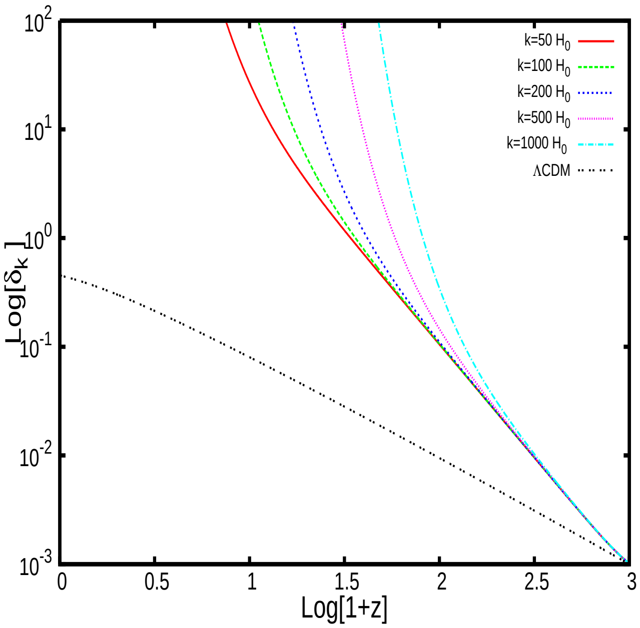

Figure 1: δ_k evolution for $f_A(R,T)$ model according to the quasi-static evolution given by (40) and ΛCDM given by (39). The depicted modes are $k = 50$, 100, 200, 500 and 1000 (in H_0 units). The plotted redshift ranged from $z = 1000$ to $z = 0$ (today). The value of Ω_m^0 was fixed to 0.27 for illustrative purposes. It is seen how whereas the ΛCDM is k-independent, the $f_A(R,T)$ model evolutions diverge for all the studied modes and leave the linear region at redshifts $z \approx 100$. For larger k-modes (deep Sub-Hubble modes) the divergence happens at larger redshift (earlier in the cosmological evolution).

A. $f_A(R_0, T_0) = R_0 + \alpha T_0^{1/2}$

This model encapsulates a modification to GR purely originated by the function $f_2(T_0)$ introduced in Section 2 through the expression (19).

For this function we parametrize the constant α as follows from (43) that possesses the appropriate dimensions and where c_1 holds for a non-dimensional constant. For this model, one can solve the background evolution

$$\tilde{\mathcal{H}}^2 \;=\; \Omega_m^0 a^{-1} + (1-\Omega_m^0) a^{1/2} \tag{47}$$

numerically where we are using dimensionless time defined as $\tilde{\eta} = H_0\eta$ and Ω_m^0 holds for the usual fractional matter density today. According to the last equation the parameter c_1 must accomplish

$$c_1 \;=\; -\frac{1-\Omega_m^0}{\Omega_m^0}\,, \tag{48}$$

in order to satisfy (47).

For this model, we compare our result (40) with the standard ΛCDM quasi-static approximation (39). The initial conditions are given at redshift $z = 1000$ where δ is assumed to behave as in a matter dominated universe, i.e. $\delta_k(\eta) \propto a(\eta)$ with no k-dependence.

In Fig. 1 we have plotted the evolution of the density contrast for several modes. One can see how the strong k-dependence of equation (40) renders the evolution of these modes completely incompatible with the density contrast evolution provided by the Concordance ΛCDM model and leads δ outside the linear order at redshift $z \approx 100$.

B. $f_B(R_0, T_0) = R_0 + \alpha T_0^{1/2} - 2\beta$

Let us now consider the general model found in (19), which also satisfies the usual continuity equation in the background but where the usual GR term is supplemented with a cosmological constant -2β. The first FLRW equation (14) yields,

$$\tilde{\mathcal{H}} = \Omega_m^0 a^{-1} - c_1\Omega_m^0 a^{1/2} + c_2 a^2\;, \tag{49}$$

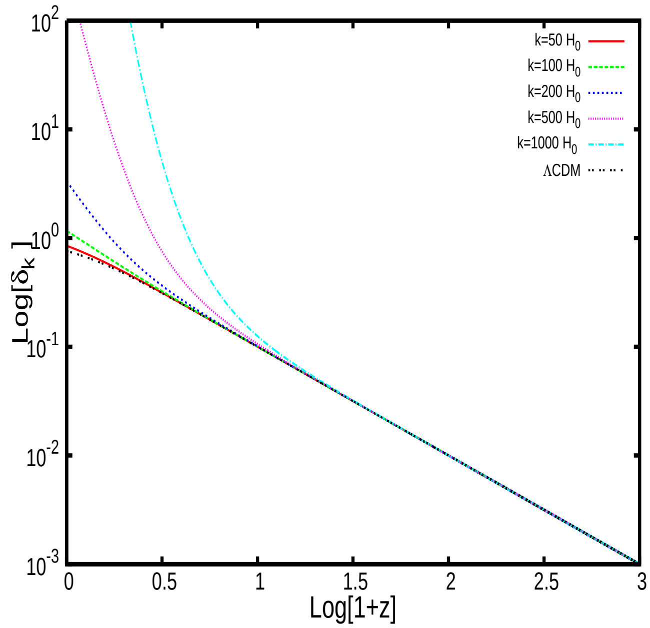

Figure 2: δ_k evolution for $f_B(R,T)$ model according to the quasi-static evolution given by (37) and ΛCDM given by (39). Here we have assumed a value $c_1 = -10^{-3}$. As previously, the dependence on k leads to a strong growth of the matter perturbations for large values of k, whereas the behavior is similar to the ΛCDM model for the modes $k < 200H_0$.

where we have (43), and $\beta = 3H_0^2 c_2$ in order to provide the correct dimensions to the free constants parameters $\{\alpha,\ \beta\}$. We have also rewritten the Hubble parameter as $\mathcal{H} = H_0\tilde{\mathcal{H}}$, where H_0 is the value today, and have defined $\Omega_m^0 \equiv \kappa^2 \frac{\rho_m(\eta_{today})}{3H_0^2}$. By assuming $a(\eta_{today}) = 1$ $(z = 0)$, the equation (49) must satisfy,

$$1 = \Omega_m^0 - c_1\Omega_m^0 + c_2 \quad \rightarrow \quad c_2 = 1 - \Omega_m^0(1 - c_1)\ . \tag{50}$$

This expression provides a constraint on the dimensionless parameters $\{c_1, c_2\}$, where one remains arbitrary. Thus, one can explore the scalar cosmological perturbations at the quasi-static approximation through the equation (37). As for the previous case, the strong dependence on k in the equation (37) leads to an evolution of the matter perturbations incompatible with the observations. In fact, only a very restricted limit for the free parameter c_1 can avoid such strong violations together with an upper limit on k. In Fig. 2, the case for a negative $c_1 = -10^{-3}$ is considered, yielding a similar behavior as in Fig. 1. Another illustrative example of the behavior of the equation (37) is shown in Fig. 3, where $c_1 = 10^{-3}$ is set. For this case, it is straightforward to check that the equation (37) turns out the damped oscillator equation for large $k - modes$, since the k dependent term is positive and dominates over the other terms for small redshifts.

VII. CONCLUSIONS

In this work we have studied the evolution of matter density perturbations in $f(R,T)$ theories of gravity. We have presented the required constraint to be satisfied by these theories in order to guarantee the standard continuity equation for the energy-momentum tensor. This constrain restricts severely the form of $f(R,T)$ models able to theoretically preserve both BBN abundances and the usual behavior of both radiation and matter as cosmological fluids.

For models of the form $f_1(R)+f_2(T)$ we have determined the unique $f_2(T) \propto T^{1/2}$ model able to obey the standard continuity equation.

Once the background evolution was imposed such viability condition, we have obtained the quasi-static approximation for these theories and shown that for sub-Hubble modes, the density contrast obeys a second order differential equation with explicit wavenumber dependence and subsequent strong divergences on the cosmological evolution of the perturbations.

This fact is in contrast with well-known results for $f(R)$ fourth order gravity theories and also Hilbert-Einstein action with a cosmological constant.

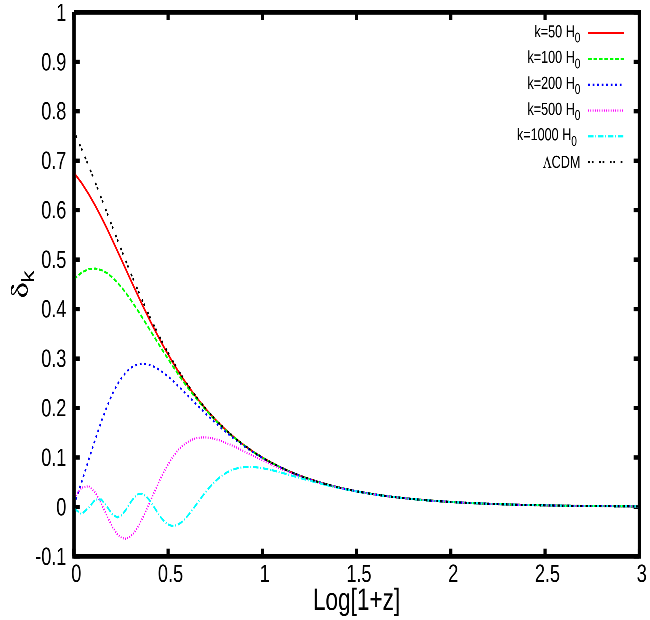

Figure 3: δ_k evolution for $f_B(R,T)$ model according to the quasi-static evolution given by (37) and ΛCDM given by (39). Here we have assumed a positive value for the free parameter $c_1 = 10^{-3}$, which leads to an oscillating behavior of the matter perturbations, which turns out stronger as k is larger, and whose oscillations are observed for large small redshifts. The model mimics the ΛCDM model only those modes small enough $k < 50H_0$.

We have then compared our results with the usual quasi-static approximation in general relativity and shown how both density contrast quantities evolve differently. Also the departure from the linear regime happens for these theories. These results are in strong contradiction with the usually assumed behavior of the density contrast and therefore sets strong limitations to the viability of these theories.

Alternatively, the study of a positive coupling constant for the modified term $T^{1/2}$ leads to a damped harmonic oscillator for large k-modes, as we illustrated in the second model under consideration, in particular in Fig. 3. Moreover, as analyzed above, the quasi-static approximation equation contains a singular point that may force the matter perturbations to diverge along the cosmological evolution. This assumption provides a way to constraining the value of the coupling constant c_1, but does not prevent the strong deviation of the sub-Hubble models for this kind of models. Hence, we conclude that the kind of models studied in this investigation, $f(R,T) = f_1(R) + f_2(T)$, where the only viable $f_2(T)$ function is given by (19), leads to a catastrophic behavior for the sub-Hubble modes in the study of matter perturbations, preventing this kind of models to be considered as a competitive candidate for dark energy.

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